Tracking the proportion of total factor count contributed by non-primes in a given range: a relative of the prime counting function?

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I got curious about the relative contributions of primes and composites to the factors of integers, and investigated this relationship with some computational experiments. The following is a description of my observations and findings. Let $F(x)$ be the total number of factors counted all the other integers in the range, then summed up the tallies for a given value of x and divided the result by the total number of factors contributed by the non-prime integers only in the same range. The result was a function $G(x)$, which I nicknamed the "factor credit" function because it expresses the relative credit due to the composite numbers in explaining factors to the integers in the given range (see Table 1).

Plotting $G(x)$ for $1 \le x \le 3000$ revealed a sharp $\sqrt{x}$ transition rate to a sudden peak at $(7, 1.57777...)$ followed by a gradual stabilizing and decay (Figure 1). The function's overall shape suggested a damped oscillation around a horizontal asymptote. Further experiments revealed a transition related to $\ln(x) / \pi(x)$. I plotted translations of a variation function on the same graph and they appeared to provide reasonably good upper and lower bounds for $G(x)$ in the chosen interval, but I was unable to extend the investigations to a larger interval. The horizontal asymptotes of these translated rational functions were located at $y = 1.39939$ and $y = 1.16939$, and I wondered if $G(x)$ might converge on a value between these, but I was unsure.

It seems like this must be related to the prime counting function...maybe a variation of the same? I get the impression that what it's counting is connected to that, but the exact nature of the connection isn't clear to me; I'm curious to learn more, such as the formal name for the function, whether it does converge (and to what value it is), and what kind of regression provides the best fit for the curve.

													$S(x)$	$S_{comp}(x)$	$G(x)$
		1	2	3	4	5	6	7	8	9	10	11			
	1	1											1	1	1
	2	2	1										3	2	1
	3	3	1	1									5	3	1.66666...
	4	3	2	1	1								8	5	1.6
x	5	4	2	1	1	1							10	6	1.66666...
	6	5	3	2	1	1	1						14	8	1.75
	7	6	3	2	1	1	1	1					16	9	1.77777...
	8	7	4	2	2	1	1	1	1				20	12	1.66666...
	9	8	4	3	2	1	1	1	1	1			23	14	1.64285...
	10	9	5	3	2	2	1	1	1	1	1		27	16	1.6875

TABLE 1. Contribution of an integer n in the range $[1, x]$ to the total number of factors of all integers in the range. Values for $1 \le x \le 10$ are shown. The rightmost columns give the total factor count $S(x)$ for all integers in the range $[1, x]$, the smaller factor count $S_{comp}(x)$ contributed by non-primes only in the same range, and the ratio $G(x) = S(x) / S_{comp}(x)$.

FIGURE 1. Scatterplot of "factor credit" function $G(x)$ for $1 \le x \le 3000$, bounded above and below by vertical translations f_U and f_L of a rational function $f(x) = ((1.4988 * (x + 1.32) + 1.12 * x (0.98 * x + 1.32) + 0.088 * x + 1.1))$. The horizontal asymptotes of the two translations are shown by the dashed lines. The function $G(x)$ is undefined for $x = 1$ because the denominator of the fraction is zero. Vertical translations of $f(x)$ by $\pm 6/\pi$ have been summed with the 300 adjacent values immediately before and after $f(x)$ and the averages of those taken. The maximum of $|f_U(x) - G(x)|$ occurs at $x = 2700$ and the minimum at $x = 605$, while the maximum of $|f_L(x) - G(x)|$ occurs at $x = 600$ and the minimum at $x = 2700$, suggesting $G(x)$ is slowly moving away from $f_L(x)$ and toward $f_U(x)$. All x-values have been natural log transformed for ease of viewing (see scale on x-axis).

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Appendix

Values of $G(x)$ for $1 \le x \le 3000$